Tarski's relation algebra and Börners operator

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In 1975, Alfred Tarski visited Unicamp-Brazil. In that ocasion, an event was organized to welcome him, *Simpósio de Lógica Matemática*. Tarski delivered two conferences about his theory on the calculus of binary relations, the so called *relation algebra* (RA). He introduced some notions and fundamental results, concerning RA, along with some open problems. His view on these problems influenced research groups that continued studying RA and have developed relevant work on this area.

The first open problem introduced by Tarski, which will be analised in this work, was concerned to the foundation of RA. We start from de definition of RA so that we can introduce the problem.

Definition 0.1. (a) A *relational algebra* is a structure

$$\mathfrak{A} = \langle A, +, {}^{-}, ;, {}^{\cup}, 1' \rangle,$$

in which A is a no empty set, $+$ and $;$ are binary operations in A, ${}^{-}$ and ${}^{\cup}$ unary operations in A, and $1'$ is a distinguished element in A.
(b) A *relation algebra* is a relational algebra that satisfies the following postulates, for

all $a, b, c \in A$:

$$P1) \quad a + b = b + a;$$

$$P2) \quad a + (b + c) = (a + b) + c;$$

$$P3) \quad (a^{-} + b)^{-} + (a^{-} + b^{-})^{-} = a;$$

$$P4) \quad a ; (b ; c) = (a ; b) ; c;$$

$$P5) \quad a ; 1' = a;$$

$$P6) \quad a^{\cup\cup} = a;$$

$$P7) \quad (a ; b)^{\cup} = b^{\cup} ; a^{\cup};$$

$$P8) \quad a ; b + c^{\cup -} = c^{\cup -} \;\Rightarrow\; b ; c + a^{\cup -} = a^{\cup -}.$$

(c) RA is the class of all relation algebras.

It can be seen in the definition of RA that, as it was done, its structure contains five primitive operators. Some questions about it can be raised, would it be possible to have less operators to define RA? How far can we go in decreasing the number of operators in the definition so that we can keep RA's definition in terms of equations only?[1] Now we quote Tarski to approach the problem:

" *Can we decrease the number of these primitive notions so that we still have a variety, a class of algebras defined by means of equations only?...Now, if you have only two binary operations, the big problem is if you can replace these two binary operations by one binary operation. Can you treat RA as group always? As algebra with one binary operation only, but to be able to characterize them again with postulate system consisting only of equations? I suspect that the answer to this problem is negative, but I have no proof of it.*"

It's interesting to notice that Tarski's suspecting about the impossibility to find such an operator was mistaken, since it was showed by Ferdinand Börner, in 1986, that it's possible to find an binary operator which can replace all the other operators used to define RA.

Börner used the notion of *clones* to construct such an operator. Even though this operator was constructed in a quite artificial way and it is not intuitive at all, Börners

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[1] Here, in our definition of RA we don't have only equations but it can be shown that this definition is equivalent to a equational definiton of RA, which is a important fact to prove that RA is a *variety*

work has a intrinsic value for its intelectual insight. In what follows, we show how Börner "answers" Tarski's problem.

Let $Re(U)$ be the set of all binary relations on a set U (U must have at least two elements) and $Op^{(n)}$ the set of all n-ary operations on $Re(U)$, tha is,

$$Op^{(n)}(Re(U)) = \{f/f : Re(U)^n \rightarrow Re(U)\} \quad (n \in \mathbb{N}^+)$$

and, moreover,

$$Op(Re(U)) = \bigcup_{n \in \mathbb{N}} Op^{(n)}(Re(U))$$

denotes the set of all operations on $Re(U)$. For $F \subset Op(Re(U))$, put $F^{(n)} = F \cap Op^{(n)}(Re(U))$.

For $i, n \in \mathbb{N}, 1 \leq i \leq n$, pr_i^n denotes the n-ary *projection* to the i-th argument:

$$pr_i^n(R_1, ..., R_n) = R_i.$$

For $f \in Op^{(n)}(Re(U)), g_1, ..., g_n \in Op^{(m)}(Re(U))$, the operation $f(g_1, ..., g_n)$, defined by

$$f(g_1, ..., g_n)(R_1, ..., R_m) = f(g_1(R_1, ..., R_m), ..., g_n(R_1, ..., R_m))$$

is called the *superposition* of f and $g_1, ..., g_n$.

Definition 0.2. A set $F \subset Op(Re(U))$ is called a *clone over* $Re(U)$, if the following conditions hold:
i) F contains all projections;
ii) F is closed under superposition, i.e,

$$f \in F^{(n)}, g_1, ..., g_n \in F^{(m)} \rightarrow f(g_1, ..., g_n) \in F^{(m)}.$$

The intersection of clones is again a clone. Thus we can define:

Definition 0.3. For a set $G \subset Op(Re(U))$ let

$$\text{clone}_{Re(U)} G = \bigcap \{F | G \subset F \wedge F \text{is clone over} Re(U)\}$$

be the smallest clone that contains G. We say G is a *generating set* for $\text{clone}_{Re(U)} G$ (or G generates $\text{clone}_{Re(U)} G$).

A clone F is called *finitely generated*, if there exists a finite generating set and F is called *k-generated* ($k \in \mathbb{N} = \{0,1,2,...\}$), if there exists a generating set with at most k elements.

In Boolean algebras, we know an operator that replaces the operations of union, intersection, complement, unit and empty. Such operator is known as *Shefer stroke*, denoted by $|_{Sh}$ and defined as $R|_{Sh}S = R^{-}.S^{-}$, where "$\cdot$" stands for Boolean intersection.

From the fact just mention aboved, we know that a *Boolean clone* $C_B(U) = \text{clone}_{Re(U)}\{+\ ,\ .\ ,\ ^{-},\ 0\ ,\ 1\}$ is one generated, e.g., $C_B(U) = \text{clone}_{Re(U)}\{|_{Sh}\}$.

The following results (like the notions introduced above) can be seen in [Börner1986].

Since Boolean algebra is a reduct of RA, the following result is essential to prove that there exist an operator that generates the operations of RA.

Teorema 0.4. *Every finitely generated clone F over* $Re(U)$ *containing the Sheffer stroke* $|_{Sh}$ *is one-generated..*

Next step, in the following theorem and its corollaries, is to determine conditions concerning the relative product (;) to show the existence of a one-generated clone that contains $|_{Sh}$ and ;.

Teorema 0.5. *Let U be a finite set and let*

$$m = |Re(U)| = 2^{|U|^2}.$$

Every clone over $Re(U)$ *that contains the Sheffer stroke and the relational composition is finitely generated by the set of its m-ary operations.*

$$;, |_{Sh} \in F \wedge F\,clone\ over Re(U) \Rightarrow F = clone_{Re(U)} F^{(m)}$$

Corolário 0.6. *If U is finite, then every clone F over* $Re(U)$ *that contains* $|_{Sh}$ *and ";"is one generated.*

Corolário 0.7. *If U is finite, then there exist only a finite number of clones over* $Re(U)$ *that contains the Sheffer* $|_{Sh}$ *and the relational composition ";".*

The results obtained so far are suficient to point out Tarski's "mistake" in suspecting that it wouldn't be possible to find an operator which could, by itself, define RA. But, of course, for the answer to be complete it is important to show what this operator is like.

When Tarski introduced the problem, he asked if it would be possible to treat RA as *group* always. Roger Maddux understand that what Tarski meant was if it would be possible to have one, and only one, **binary** operator, as we have in groups. It's important to remember that Hajnal Andréka had showed, before Börner's article, one operator which could replace all others, but it wasn't a binary one. The following theorem completes Börner's answer to Tarsk's problem.

Teorema 0.8. *The clone of every relation algebra is generated by the binary operation*

$$h(x,y) = (x^- \cap y^-) \cup (g_1(x,y); g_2(x,y); (g_3(x,y) \cup g_4(x,y)))$$

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$g_1(x,y) = (x \cap y^- \cap E^-)^\cup$
$g_2(x,y) = (x \cup y) \cap E$
$g_3(x,y) = (V; (x \cap y \cap E); V) \cap g_1(x,y)$
$g_4(x,y) = (V; (x \cap y \cap E); V)^- \cap (((V; (y \cap E); V) \cap (x^- \cap y^- \cap E^-)^\cup) \cup ((V; (y \cap E); V)^- \cap E)))$.

Note that the results demonstrated by Börner hold for finite RAs. Tarski could be interested in an operator which could be used for any RA, including infinite ones, however, he never mentioned any restriction for the number of elements of the universe that a RA could have so that the operator could be found, that makes us think that even with Börner's restriction, this last theorem would "surprise" Tarski.

It seems that [Börner1986] wasn't taken to its extent to see how much such an operator could change the studies of RA. The implications of Börner's operator for the development of RA may still need a carefull look. The fact is that the research about RA has ramified in so many different branches that something may have been left along the way, however, even if it is not the case for Börner's results, this work preserves a historical content about RA and its "creator".

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